Chapters 5.1 Trees

An edge e in a connected graph G is a **bridge** if $G - e$ is not connected.

An edge e in a graph G is a **bridge** if the number of connected components in $G - e$ is more than in G.

1: An edge e of a graph G is a bridge if and only if e lies on no cycle of G.

Solution: If e is in a cycle, then e is not a bridge since removing still allows walks. If $e = uv$ is not in a cycle, then there is no path P between its endpoints (if there was one, $P + e$ would be a cycle), hence in $G - e$ the number of connected components increase.

A graph is **acyclic** if it has no cycles.

A graph G is a **tree** if G is acyclic and connected.

2: List all non-isomorphic trees on 6 vertices.

Solution:

End-vertex or **leaf** is a vertex of degree one.

A tree is a **star** if it has exactly one vertex that are not a leaf.

A tree is a **double-star** if it has exactly two vertices that are not leaves.

A tree is G a **caterpillar** if G has at least 3 vertices and removing all leaves from G gives a path, the path is called **spine** of the caterpillar.

An acyclic graph is called a **forrest**.

3: List all non-isomorphic forests on 5 vertices.

Solution:

4: Draw a star, a double star, and a caterpillar on 7 vertices. Are any of these unique?

Solution:

Lemma 5.1.3 Every tree with at least two vertices has at least two end-vertices.

5: Prove the lemma. *Hint: Take longest path.*

Solution: Let G be any tree on at least 2 vertices. Let P be the longest path in G. Its endpoints must have degree 1. If not, the path can be extended as G has no cycles.

Lemma 5.1.3 (Tree-growing lemma) Let G be a graph and v is end-vertex. Then G is a tree if and only if $G - v$ is a tree.

6: Prove the lemma, both directions.

Solution: Assume that G is a tree. The G has no cycles and hence $G - v$ has no cycles. Now we need to show that $G - v$ is connected. Let x, y be two vertices i $G - v$. There is a path P in G with x, y being endpoints. Since v has degree 1, it is not in P. Hence P is in $G - v$ and $G - v$ is connected. Hence it is a tree.

Assume $G - v$ is a tree. By adding a vertex v, it remains connected and adding a vertex of degree 1 does not create a cycle.

Theorem 5.1.1 For a graph $G = (V, E)$, the following are equivalent.

1. G is a tree.
2. (Path uniqueness) Every two vertices of G are connected by a unique path.
3. (Minimal connected) G is connected and for every edge $e \in E$, $G - e$ is disconnected.
4. (Maximal without cycle) G has no cycles and for every $x, y \in V$ such that $xy \notin E$, $G + xy$ contains a cycle.
5. (Euler's formula) G is connected and $|V| = |E| + 1$.

7: Show that 1 implies all of 2, 3, 4, 5.

Solution:

8: Show 2 implies 1, i.e. if every two vertices of G are connected by a unique path, then G is a tree.

Solution: Right from the beginning, G is connected. If G had a cycle, there would be no uniqueness of the path. And that is it.

9: Show 3 implies 1, i.e. if G is connected and for every edge $e \in E$, $G - e$ is disconnected, then G is a tree.

Solution: Right from the beginning, G is connected. If G had a cycle C, then $G - e$ would be still connected for any $e \in E(C)$.

10: Show 4 implies 1, i.e. if G has no cycles and for every $x, y \in V$ such that $x, y \notin E$, $G + e$ contains a cycle, then G is a tree.

Solution: Right from the beginning, G has no cycles. If G had two connected components, then adding any edge with endpoints in both of these 2 components create no new cycle.

11: Show 5 implies 1, i.e. if G is connected and $|V| = |E| + 1$, then G is a tree.
Hint: Show G has an end-vertex.

Solution: Right from the beginning, G is connected. If every vertex had degree at least 2, then by the hand-shake lemma

$$2|E| = \sum_v d(v) \geq \sum_v 2 = 2|V|.$$

So $|E| \geq |V|$, which contradicts that $|V| = |E| + 1$. Let v be a leaf. Let G' be $G - v$. Observe that G' satisfies $|V(G')| = |E(G')| + 1$ and G' is connected. By induction, G' is a tree. And hence G is also a tree.

12: Show that a graph G satisfying $|V| = |E| + 1$ need not be a tree.

Solution: Take a disconnected graph. One cycle and the other component is a tree.

13: Prove that every n-vertex graph with m edges has at least $m-n+1$ cycles (different cycles, but not necessarily disjoint cycles).

Solution:

14: Show that if T is a tree and $\Delta(T)=k$ then T has at least k leaves. Recall that $\Delta(T)$ means the maximum degree of T.

Solution:

15: Show that sequence of natural numbers $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ is a degree sequence of some tree iff $\sum_i d_i = 2n-2$.

Solution:

16: Let G be a connected graph that has neither C_3 nor P_4 as an induced subgraph. Prove that G is a complete bipartite graph.

Solution:

17: [Open problem] In a 3-regular graph, is there always a cycle whose length is a power of 2? Is it true for the Petersen's graph?